MATH 3060 Assignment 3 solution

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October 14, 2022

1. The onlyif direction is clear, we prove the other direction. Assume d satisfies (i) and (ii), we first check $d(x,y) = d(y,x)$. Put $z = y$ in condition (ii), we get
$$d(x,y) \leq d(y,x) + d(y,y) = d(y,x).$$
Swapping the roles of x and y, we also get $d(y,x) \leq d(x,y)$, hence $d(x,y) = d(y,x)$. Having proved that d is symmetric, condition (ii) is now equivalent to the triangle inequality. This shows d is a metric.

2. (a) It is clear that $d_1(x,y) = d_1(y,x)$, $d_1(x,y) \geq 0$ and $d_1(x,y) = 0$ if and only if $x = y$. Moreover, for $x, y, z \in l_1$,
$$\begin{aligned} d_1(x,y) + d_1(y,z) &= \sum(|x_i - y_i| + |y_i - z_i|) \\ &\geq \sum |x_i - z_i| = d_1(x,z), \end{aligned}$$
This proves the triangle inequality.

   (b) It is clear that $d_2(x,y) = d_2(y,x)$, $d_2(x,y) \geq 0$ and $d_2(x,y) = 0$ if and only if $x = y$. Moreover, for $x, y, z \in l_2$. We want to show the triangle inequality:
$$\begin{aligned} & d_2(x,y) + d_2(y,z) \geq d_2(x,z) \\ \iff & \sqrt{\sum(x_i - y_i)^2} + \sqrt{\sum(y_i - z_i)^2} \geq \sqrt{\sum(x_i - z_i)^2} \\ \iff & \left(\sqrt{\sum(x_i - y_i)^2} + \sqrt{\sum(y_i - z_i)^2}\right)^2 \geq \sum[(x_i - y_i) + (y_i - z_i))]^2 \\ \iff & \sqrt{\sum(x_i - y_i)^2 \sum(y_i - z_i)^2} \geq \sum(x_i - y_i)(y_i - z_i) \\ \iff & \sum(x_i - y_i)^2 \sum(y_i - z_i)^2 \geq \left(\sum(x_i - y_i)(y_i - z_i)\right)^2 \end{aligned}$$
which is the Cauchy Schwartz inequality.

   (c) It is clear that $d_\infty(x,y) = d_\infty(y,x)$, $d_\infty(x,y) \geq 0$ and $d_\infty(x,y) = 0$ if and only if $x = y$. Now let $x, y, z \in l_\infty$, for each $i \in \mathbb{N}$, we have
$$|x_i - z_i| \leq |x_i - y_i| + |y_i - z_i| \leq d_\infty(x,y) + d_\infty(y,z)$$

Taking the supremum, we get the triangle inequality:

$$d_\infty(x,z) \le d_\infty(x,y) + d_\infty(y,z)$$

(d) Let $x \in l_1$, then since $\sum |x_i| < \infty$, we can find $n \in \mathbb{N}$ s.t. $|x_i| < 1$ for $i > n$. Then

$$\begin{aligned}\sum_{i=1}^{\infty} |x_i|^2 &= \sum_{i\le n} |x_i|^2 + \sum_{i>n} |x_i|^2 \\ &\le \sum_{i\le n} |x_i|^2 + \sum_{i>n} |x_i| \\ &\le \sum_{i\le n} |x_i|^2 + \sum_{i=1}^{\infty} |x_i| < \infty\end{aligned}$$

so $x \in l_2$.
Next assume $y \in l_2$, then we can similarly find $n \in \mathbb{N}$ s.t. $|y_i| < 1$ for $i > n$. Thus

$$\sup_{n\in\mathbb{N}} |y_i| \le \max\{1, y_1, y_2, \ldots, y_n\} < \infty$$

so $y \in l_\infty$.

3. (a)

$$\begin{aligned}|\Phi(f) - \Phi(g)| &= \left| \int_a^b \sqrt{1+f^2} - \sqrt{1+g^2} \right| \\ &\le \int_a^b \frac{|f-g||f+g|}{\sqrt{1+f^2}+\sqrt{1+g^2}} \qquad \le \int_a^b \frac{|f-g||f+g|}{|f+g|} = d_1(f,g)\end{aligned}$$

Thus Φ is continuous.

(b) Since $d_1 \le (b-a)d_\infty$, part (a) shows that

$$|\Phi(f) - \Phi(g)| \le (b-a)d_\infty(f,g)$$

so Φ is continuous.

(c) Consider the sequence of functions $f_n : [-1,1] \to \mathbb{R}$ ($n \in \mathbb{N}$) with

$$f_n(x) = \begin{cases} |1-nx|, & \text{if } x \in [-1/n, 1/n] \\ 0, & \textit{otherwise.} \end{cases}$$

Then $||f_n||_1 = \frac{1}{n} \to 0$, i.e. $d_1(f_n, 0) \to 0$, but $f_n(0) = 1$ for all n, this shows Ψ is not continuous.

(d) Ψ is continuous because $|f(0) - g(0)| \le d_\infty(f,g)$.

4. Let f_n be a sequence in $C[a,b]$ that converges to f. Suppose further $f_n \ge \alpha$, let $x \in [a,b]$, we want to show $f(x) \ge \alpha$. Note that

$$f(x) - \alpha \ge (f_n(x) - \alpha) - |f(x) - f_n(x)| \ge -d_\infty(f, f_n)$$

Taking $n \to \infty$, we get the desired result.